Note on the errors in TOF energy measurements due to irregularities in the macrobunch structure.

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1.Introduction. Time of flight (TOF) technique will be used for the beam energy measurement in SNS linac. This method is based on the precise measurement of the beam phase in respect to reference signal on the fundamental frequency of the beam (402.5 MHz). The accuracy of the phase measurement is defined by many factors: accuracy of a measuring device, precision of a phase pickup and by phase noise of the beam. One possible reason for the phase noise is irregularity in macrobunch structure. If our phase detector would be capable to resolve the phase of each individual microbunch inside the macrobunch we could calculate the energy of each individual microbunch, the mean energy of macrobunch and the energy distribution along macrobunch. But bandwidth of real phase detector is limited and we can measure the average phase of some number N of microbunches only. In this note we calculate amplitude and phase of the fundamental harmonic of an irregular sequence of finite length and estimate importance of this effect for the SNS linac.

2. Fourier spectrum of an irregular sequence. Consider a sequence F(t) of N short pulses where each pulse has the same shape f (t) but slightly different amplitude q_k . Pulses appears at the moments of time $t_k = k\cdot T+\delta t_k$, where T is constant period and δt_k is deviation of each pulse from regular periodicity. Mathematical expression for f(t) is

$$F(t) = \sum_{k=0}^{N} q_k \cdot f(t - t_k),$$

Fourier transformation of F(t) is

$$F(w) = \sum_{k=0}^{N} q_k \cdot f(w) \cdot e^{iwt_k} = \sum_{k=0}^{N} q_k \cdot f(w) \cdot e^{iw(kT+dt_k)} = f(w) \cdot \sum_{k=0}^{N} q_k \cdot e^{iwkT} \cdot e^{iwdt_k},$$

Where f(ω) is Fourier transformation of f(t). Then fundamental harmonic defined by the condition $\omega_0 T = 2\pi$ can be found:

$$F(w_0) = f(w_0) \cdot \sum_{k=0}^{N} q_k \cdot e^{iw_0kT} \cdot e^{iw_0dt_k} = f(w_0) \cdot \sum_{k=0}^{N} q_k \cdot e^{i2pk} \cdot e^{iw_0dt_k} = f(w_0) \cdot \sum_{k=0}^{N} q_k \cdot e^{iw_0dt_k}.$$

Suppose that $q_k = q_0 (1+ \delta q_k)$, where δq_k is small dimensionless deviation from constant amplitude q_0, and define $\omega_0 \delta t_k = \delta\varphi_k$, then

$$F(w_0) = f(w_0) \cdot q_0 \cdot \sum_{k=0}^{N} e^{idj_k} + f(w_0) \cdot q_0 \cdot \sum_{k=0}^{N} dq_k \cdot e^{idj_k} \; .$$

Assuming $\delta\varphi_k << 1$ we can substitute $e^x = 1 + x + 1/2x^2$, then

$$F(w_0) = f(w_0) \cdot q_0 \cdot \sum_{k=0}^{N} (1 + idj_k - \frac{1}{2} dj_k{}^2) + f(w_0) \cdot q_0 \cdot \sum_{k=0}^{N} dq_k \cdot (1 + idj_k - \frac{1}{2} dj_k{}^2) =$$
$$= f(w_0) \cdot q_0 \cdot ((N+1) + i \cdot \sum_{k=0}^{N} dj_k - \frac{1}{2} \sum_{k=0}^{N} dj_k{}^2 + \sum_{k=0}^{N} dq_k + i \cdot \sum_{k=0}^{N} dq_k \cdot dj_k - \frac{1}{2} \sum_{k=0}^{N} dq_k \cdot dj_k{}^2) =$$
$$= f(w_0) \cdot q_0 \cdot (N+1) \cdot (1 + \frac{\sum_{k=0}^{N} dq_k - \frac{1}{2} \sum_{k=0}^{N} dj_k{}^2 - \frac{1}{2} \sum_{k=0}^{N} dq_k \cdot dj_k{}^2 + i \cdot (\sum_{k=0}^{N} dj_k + \sum_{k=0}^{N} dq_k \cdot dj_k)}{N+1}) =$$
$$= f(w_0) \cdot q_0 \cdot (N+1) \cdot (1 + \overline{dq_k} - \frac{1}{2} \overline{dj_k{}^2} - \frac{1}{2} \overline{dq_k \cdot dj_k{}^2} + i \cdot (\overline{dj_k} + \overline{dq_k \cdot dj_k})),$$

where "$\overline{}$" means averaging. Note that $F_0 = f(\omega_0) \cdot q_0 \cdot (N+1)$ is amplitude of the fundamental harmonic of unperturbed sequence of equal pulses, then deviation from unperturbed harmonic δF can be calculated as:

$$dF = \frac{F - F_0}{F_0} = \overline{dq_k} - \frac{1}{2} \overline{dj_k{}^2} - \frac{1}{2} \overline{dq_k \cdot dj_k{}^2} + i \cdot (\overline{dj_k} + \overline{dq_k \cdot dj_k}) \, .$$

Module of δF gives deviation of amplitude from unperturbed fundamental harmonic and phase of δF gives phase shift due to irregularities.

$$|dF| = \sqrt{\left(\overline{dq_k} - \frac{1}{2} \overline{dj_k{}^2} - \frac{1}{2} \overline{dq_k \cdot dj_k{}^2}\right)^2 + \left(\overline{dj_k} + \overline{dq_k \cdot dj_k}\right)^2}$$

$$dq = \arctan(\frac{\overline{dj_k}}{|dF|} + \frac{\overline{dq_k \cdot dj_k}}{|dF|}) \approx \frac{\overline{dj_k}}{|dF|} + \frac{\overline{dq_k \cdot dj_k}}{|dF|}$$

To calculate actual phase errors we have to know distributions of δq_k and $\delta\varphi_k$ but we can make some important conclusions from general properties of δq_k and $\delta\varphi_k$.

1. If $\delta\varphi_k$ has zero average value then the first member in r.h.s. vanishes and phase error is defined by correlation between δq_k and $\delta\varphi_k$ which is of the second order.

2. If $\delta\varphi_k$ has zero average value and there is no correlation between δq_k and $\delta\varphi_k$, then irregularities in the sequence of pulse do not produce phase errors in the first order.

It means that small random errors in microbunch positions and amplitudes do not produce phase noise in the first order and we have to analyze only effects leading to regular deviations in $\delta\varphi_k$ with nonzero average or effects leading to correlation between microbunch position and amplitude.

2. Phase error due to energy spread inside the macrobunch. . In fact TOF measurement provide us not an energy but velocity of the bunch. Relation between velocity and energy is not linear, therefore if we measure average velocity of the bunch and calculate energy using expression for single particle, the result is not equal to actual mean energy of the bunch.

Assume that there is an energy distribution inside the macrobunch. Energy of each microbunch can be represented as:

$$g_k = g_0 + dg_k \, ,$$

where $\overline{dg_k} = 0$. Then $g_0 = \overline{g_k}$ is average energy of the bunch. Energy spread leads to measured phase spread:

$$dj_k = j_k - j_0 = \frac{2pL}{l}\left(\frac{1}{b_k} - \frac{1}{b_0}\right) = \frac{2pL}{l}\left(\frac{g_k}{\sqrt{g_k^{\,2} - 1}} - \frac{g_0}{\sqrt{g_0^{\,2} - 1}}\right) =$$

$$= \frac{2pL}{l}\left(\frac{g_0 + dg_k}{\sqrt{(g_0 + dg_k)^2 - 1}} - \frac{g_0}{\sqrt{g_0^{\,2} - 1}}\right) \quad ,$$

where L is drift length, λ is operating wavelength of the measuring system. Assuming $\delta\gamma_k << \gamma_0 - 1$, we can expand above expression to a power series about point γ_0 to the second order:

$$dj_k \approx \frac{2pL}{l}\cdot\left(\frac{dg_k}{(g_0^2 - 1)^{\frac{3}{2}}} + \frac{3}{2}\,\frac{g_0 \cdot dg_k^{\,2}}{(g_0^2 - 1)^{\frac{5}{2}}}\right) + \ldots \, ,$$

after averaging we have:

$$\overline{dj_k} \approx \frac{2pL}{l}\cdot\left(\frac{\overline{dg_k}}{(g_0^2 - 1)^{\frac{3}{2}}} + \frac{3}{2}\,\frac{g_0 \cdot \overline{dg_k^{\,2}}}{(g_0^2 - 1)^{\frac{5}{2}}}\right) = \frac{3pL}{l}\cdot\frac{g_0 \cdot \overline{dg_k^{\,2}}}{(g_0^2 - 1)^{\frac{5}{2}}} \, ,$$

$\delta\gamma_k$ can be expressed through kinetic energy spread δW_k as

$$dg_k = (g_0 - 1)\cdot\frac{dW_k}{W_0} \quad ,$$

where W_0 is average kinetic energy of the bunch. Finally we have:

$$\overline{dj_k} \approx \frac{3pL}{l} \cdot \frac{g_0(g_0-1)^2}{(g_0^2-1)^{\frac{5}{2}}} \cdot \overline{\left(\frac{dW_k}{W_0}\right)^2} = \frac{3pL}{l} \cdot \frac{1}{b_0(1+g_0)^2} \cdot \overline{\left(\frac{dW_k}{W_0}\right)^2}$$

This phase error leads to error in calculation of mean energy:

$$\frac{\Delta W}{W_0} = \frac{3}{2} \cdot b_0^3 g_0^3 \cdot \frac{1}{g_0-1} \cdot \frac{1}{b_0(1+g_0)^2} \cdot \overline{\left(\frac{dW_k}{W_0}\right)^2} = \frac{3}{2} \cdot \frac{g_0}{1+g_0} \cdot \overline{\left(\frac{dW_k}{W_0}\right)^2} .$$

For typical value of the energy spread $\frac{\overline{dW_k}}{W_0} = 10^{-3}$, the energy error $\frac{\Delta W}{W} = 1.5 \cdot 10^{-6}$. As one can see the error is much less than energy spread in the bunch and can be neglected.

1.Conclusion. Our analysis shows that TOF energy measurements are well insensitive to irregularities in the macrobunch structure. Time irregularities as well as amplitude noise produce measured energy errors of the second order in respect to energy spread in the bunch so it can be neglected.