Bunch Shapes, Fourier Transforms, and Spectral Power Densities

R. E. Shafer 3/4/02

The three most common beam bunch shape representations are Gaussian, parabolic, and cosine-squared. A Gaussian shape is the easiest to represent in both the time and frequency domains. In general, this is the most appropriate shape for use with electron beams. For ion beams in linacs, parabolic appears to be the best representation. For high energy proton beams in circular rings, cosine-squared appears to be the preferred representation.

These three bunch shapes, all normalized to the same rms widths, are compared in both the time and frequency domains. In all three cases, analytic forms for the Fourier transform are used [1]. The time domain profiles are calculated using the Fourier cosine series expansion.

The Fourier series expansion representation of temporal profiles is perhaps most useful when frequency dependent effects must be calculated. For example the frequency response of pickups (e.g., BPMs or BCMs), correction for the low-beta effect (the "Bessel factor"), and frequency-dependent attenuation *and dispersion* in coax cables. Several examples are included.

When all three bunch profiles are normalized to the same rms width, the spectral power densities in the frequency domain are nearly identical for short bunches. For longer bunches, the spectral power density depends on both the rms width and the specific bunch profile.

================ Gaussian bunch shape ====================

$f := 402.5\cdot 10^{6} \quad w := 2\cdot\pi\cdot f$ beam bunching frequency is 402.5 MHz

$$g(n,s) := \exp\left[\frac{-n^{2}\cdot w^{2}\cdot\left(10^{-12}\cdot s\right)^{2}}{2}\right]$$

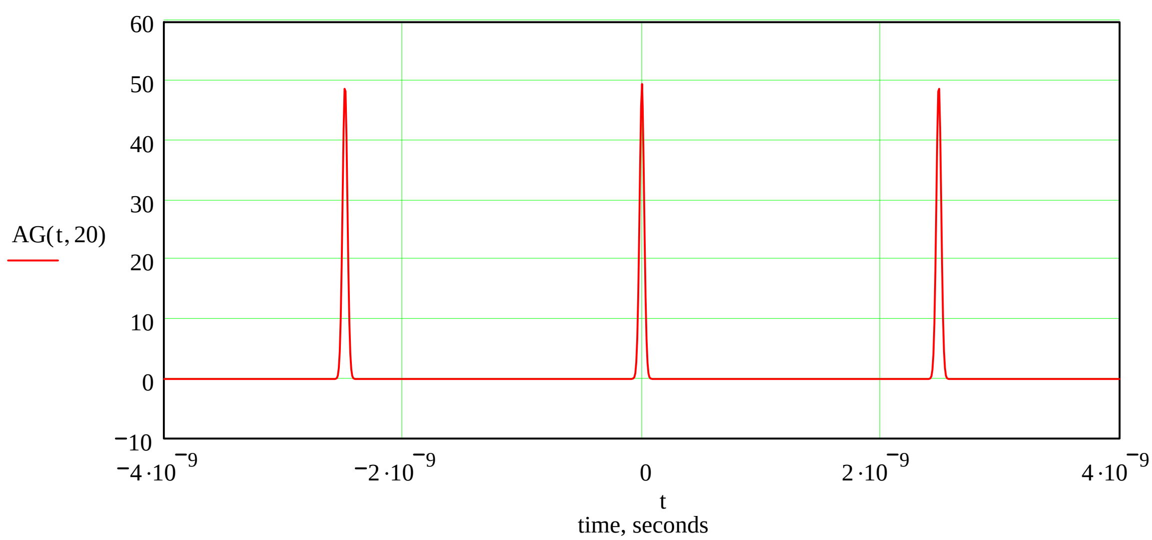
Fourier amplitude factor for Gaussian bunch

s is the rms bunch width in picoseconds

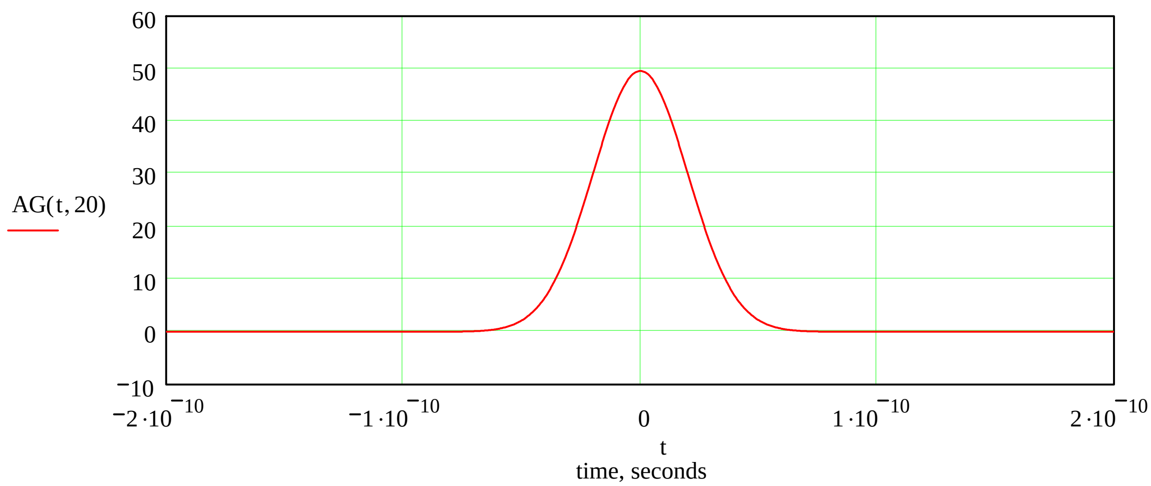
$$AG(t,s) := 1 + 2\cdot\sum_{n=1}^{100} g(n,s)\cdot\cos(n\cdot w\cdot t)$$

Fourier cosine series representation of temporal profile, with all harmonics having the same phase (assumption).

pulse train of 402.5 MHz Gaussian bunches



Temporal profile of 20-ps rms width Gaussian bunch



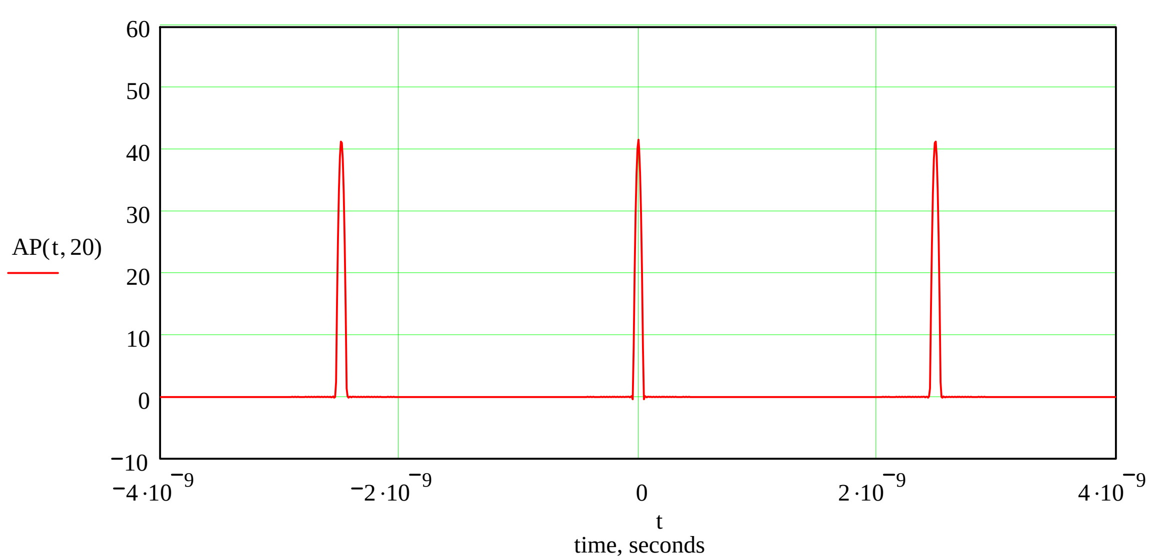
================ Parabolic bunch distribution======================

$\mathrm{alpha}(n,s) := n\cdot\pi\cdot f\cdot 4.47\cdot 10^{-12}\cdot s$ Full width at base = 4.47 rms widths, s= rms width

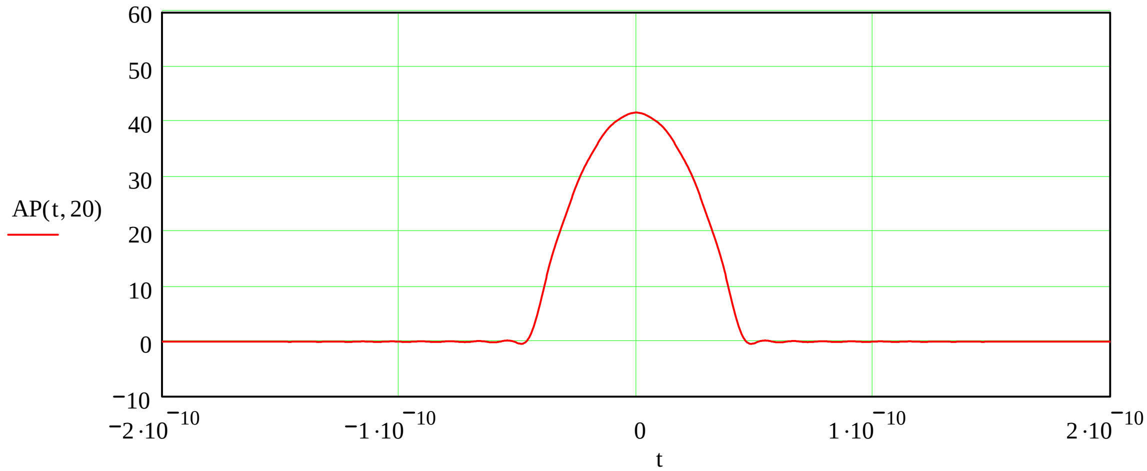
$$p(n,s) := 3\cdot\left(\frac{\sin(\mathrm{alpha}(n,s))}{\mathrm{alpha}(n,s)^3} - \frac{\cos(\mathrm{alpha}(n,s))}{\mathrm{alpha}(n,s)^2}\right)$$ Fourier amplitude factor for parabolic bunch

$$AP(t,s) := 1 + 2\cdot\sum_{n=1}^{200} p(n,s)\cdot\cos(n\cdot w\cdot t)$$ temporal spectrum of parabolic bunch

pulse train of parabolic bunches



temporal profile of a 20-ps rms width parabolic bunch



============== Cosine squared distribution ==============

$b(n,s) := 2\cdot n\cdot 5.532\cdot 10^{-12}\cdot s\cdot f$ full width at base = 5.532 rms widths, s= rms width

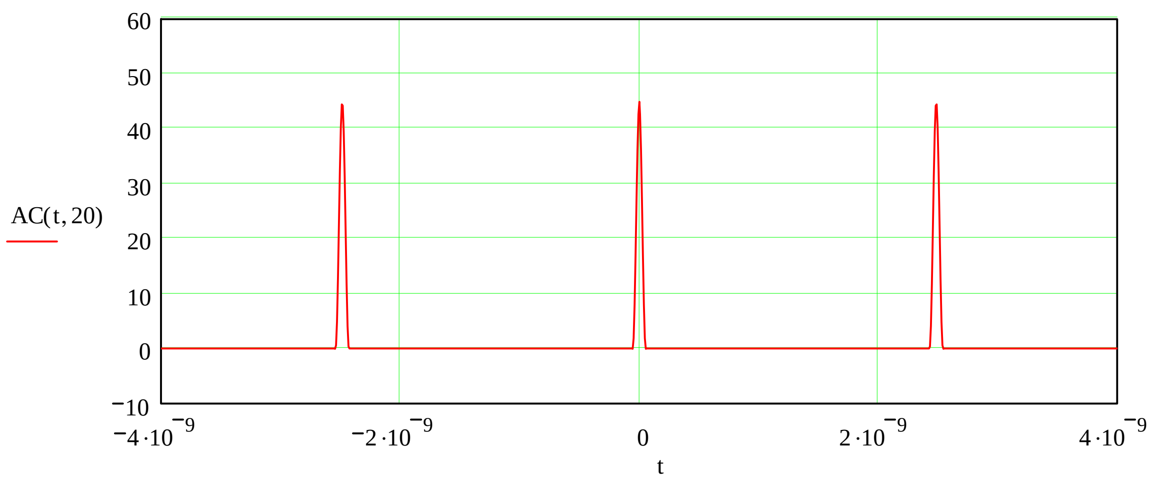
$$c(n,s) := \frac{\sin[0.5\cdot\pi\cdot(b(n,s)-2)]}{\pi\cdot(b(n,s)-2)} + \frac{\sin[0.5\cdot\pi\cdot(b(n,s))]}{0.5\cdot\pi\cdot b(n,s)} + \frac{\sin[0.5\cdot\pi\cdot(b(n,s)+2)]}{\pi\cdot(b(n,s)+2)}$$

Fourier amplitude factor for cosine-squared bunch

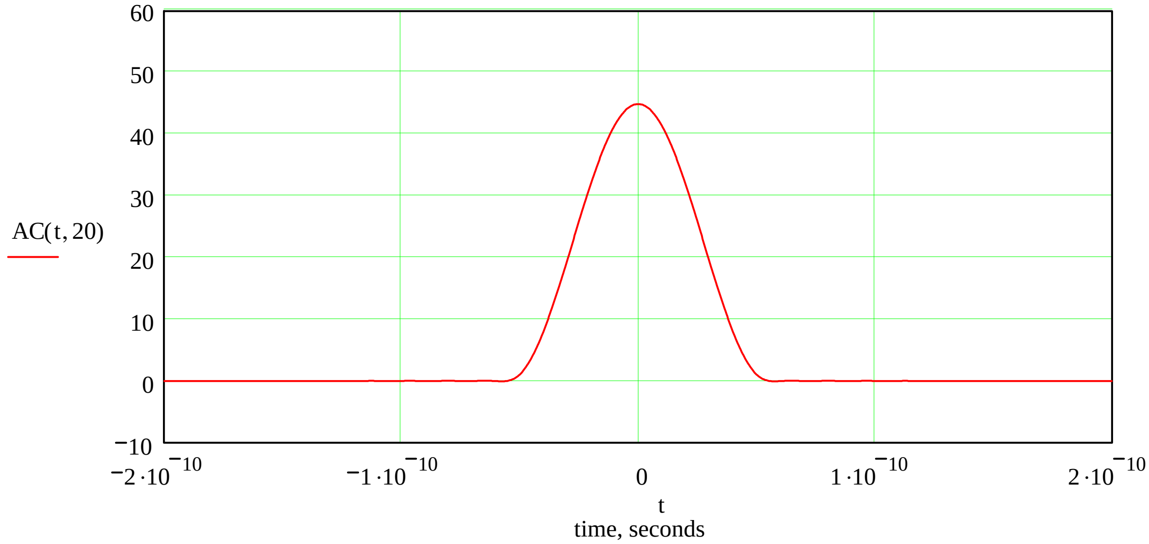
$$AC(t,s) := 1 + 2\cdot\sum_{n=1}^{150} c(n,s)\cdot\cos(n\cdot w\cdot t)$$

temporal spectrum of cosine-squared bunch

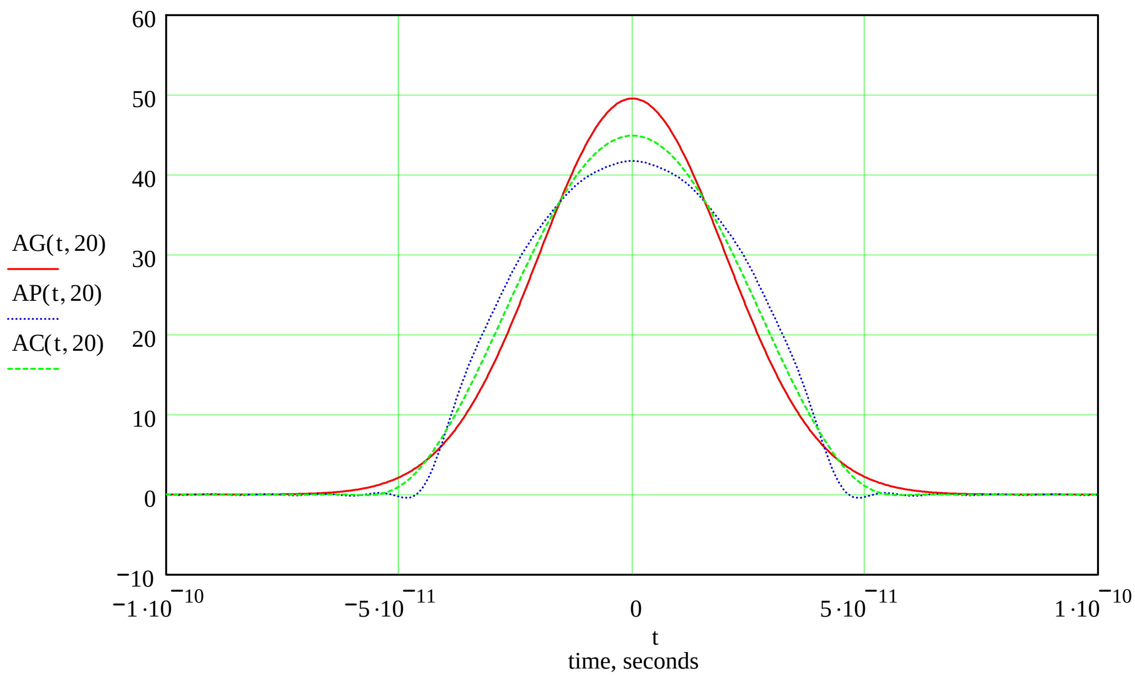
pulse train of cosine-squared bunches



temporal profile of a 20-ps rms width cosine-squared bunch



Comparison of all three bunch shapes with the same rms width (20 ps)

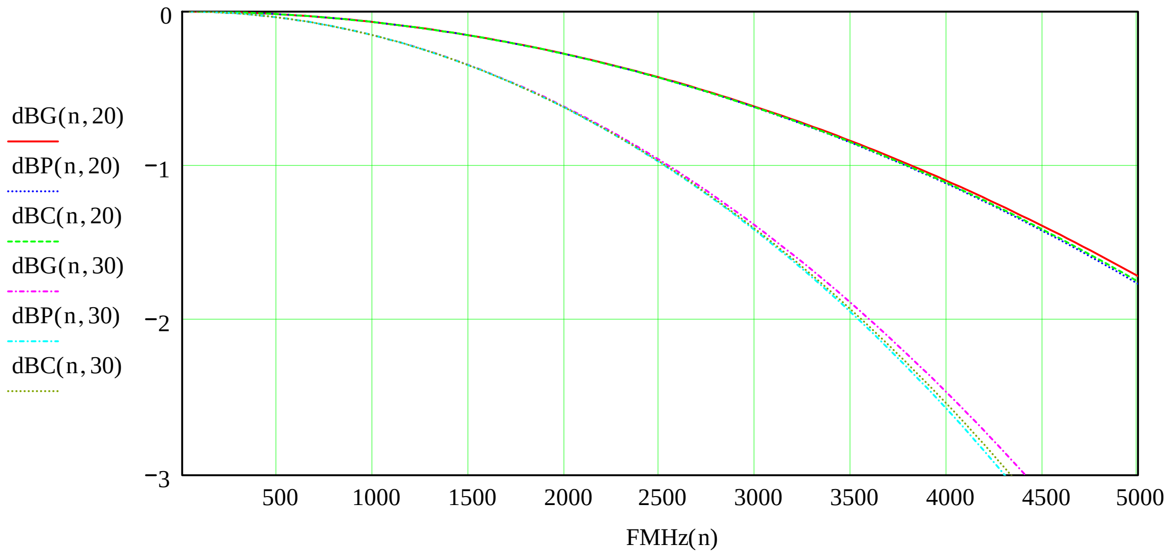


Comparison of spectral power densities up to 5 GHz for various rms bunch widths

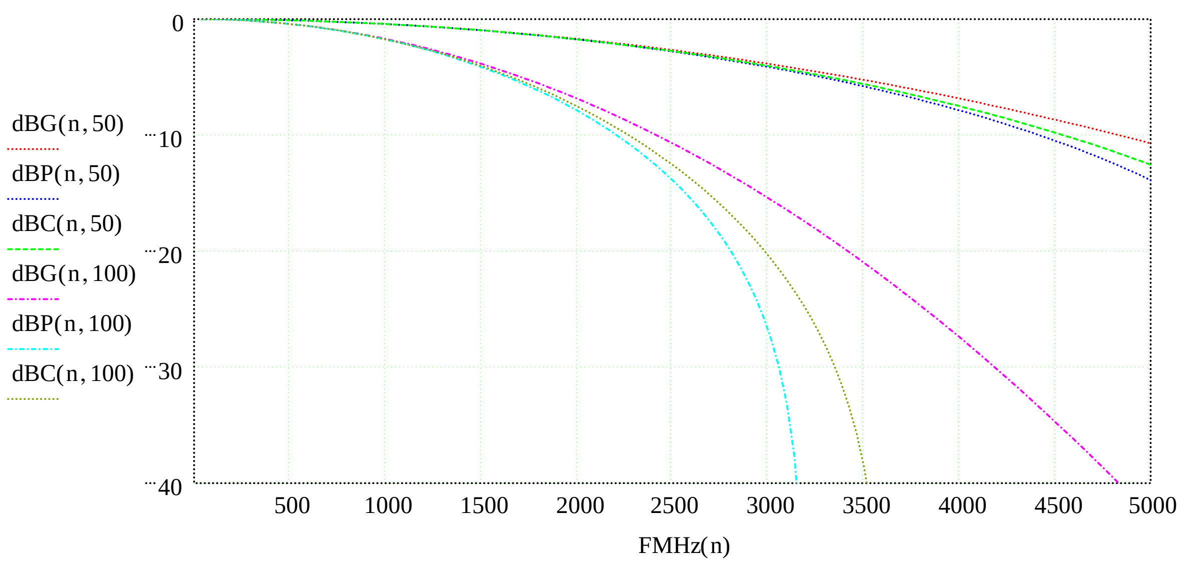
We now use the Fourier amplitude factors for the three bunch profiles to compare the spectral power densities up to 5 GHz.

$$n := 0.1, 0.2 .. 15 \qquad FMHz(n) := 10^{-6} \cdot f \cdot n$$

$$dBG(n,s) := 20 \cdot \log(g(n,s)) \qquad dBP(n,s) := 20 \cdot \log(p(n,s)) \qquad dBC(n,s) := 20 \cdot \log(c(n,s))$$



For short bunches (20 ps and 30 ps above), the spectral power density below 5 GHz is nearly independent of the bunch profile. For longer bunch lengths (50 ps and 100 ps below), the power density is very dependent on the bunch profile. For this reason, it is difficult to measure the bunch length of longer bunches using frequency-domain measurements, unless the bunch profile (e.g., Gaussian, parabolic, or cosine-squared) is known a priori. A complete spectral power density profile could determine both the bunch profile and the rms width.



======== Example of distortion caused by the low beta effect =============

$E := 300$ MeV

$c := 2.997924 \cdot 10^{10}$ cm/sec

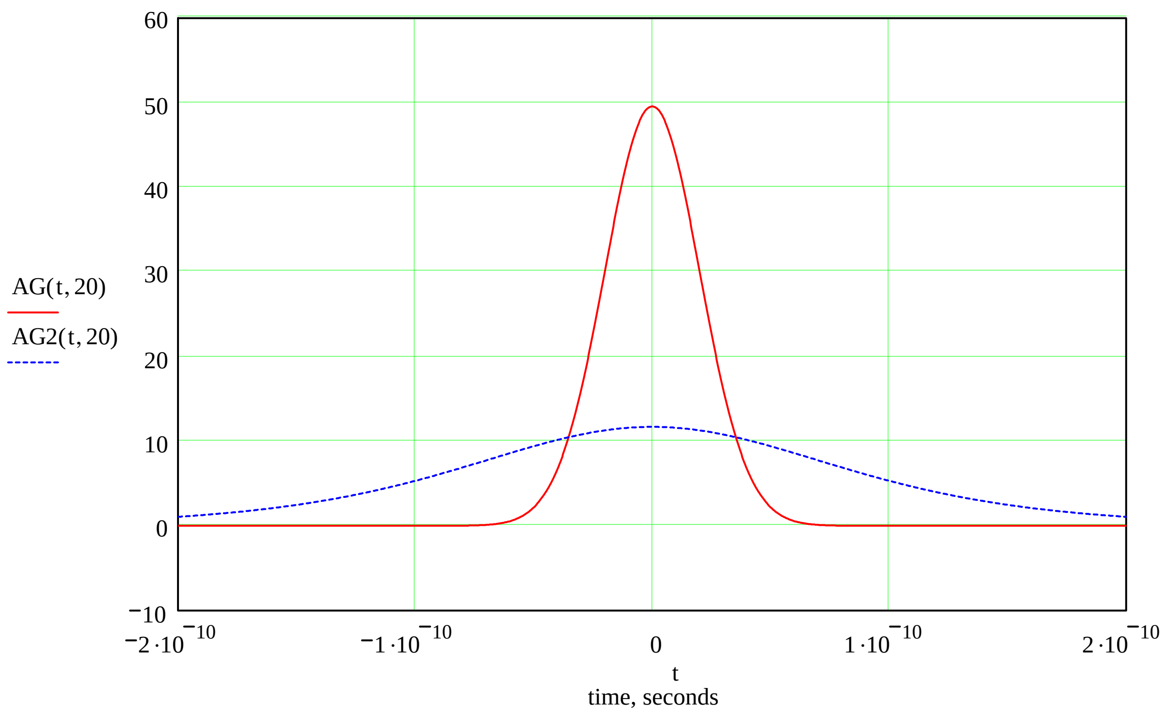
$a := 3.5$ half aperture of BPM (cm)

$$gam := 1 + \frac{E}{939.28}$$

$$bgam := \sqrt{gam^2 - 1}$$ beta*gamma

$$AG2(t,s) := 1 + 2 \cdot \sum_{n=1}^{100} \frac{1}{I0\left(\frac{n \cdot w \cdot a}{bgam \cdot c}\right)} g(n,s) \cdot \cos(n \cdot w \cdot t)$$

I_0(arg) is the modified Bessel function of order zero.



For beta = 1, the beam signal is a TEM wave, that arrives at the observation point simultaneously with the beam bunch. For beams with beta < 1, the signals are spread out over z, and arrive at the observation point before, as well as after, the beam pulse. Causality is not violated.

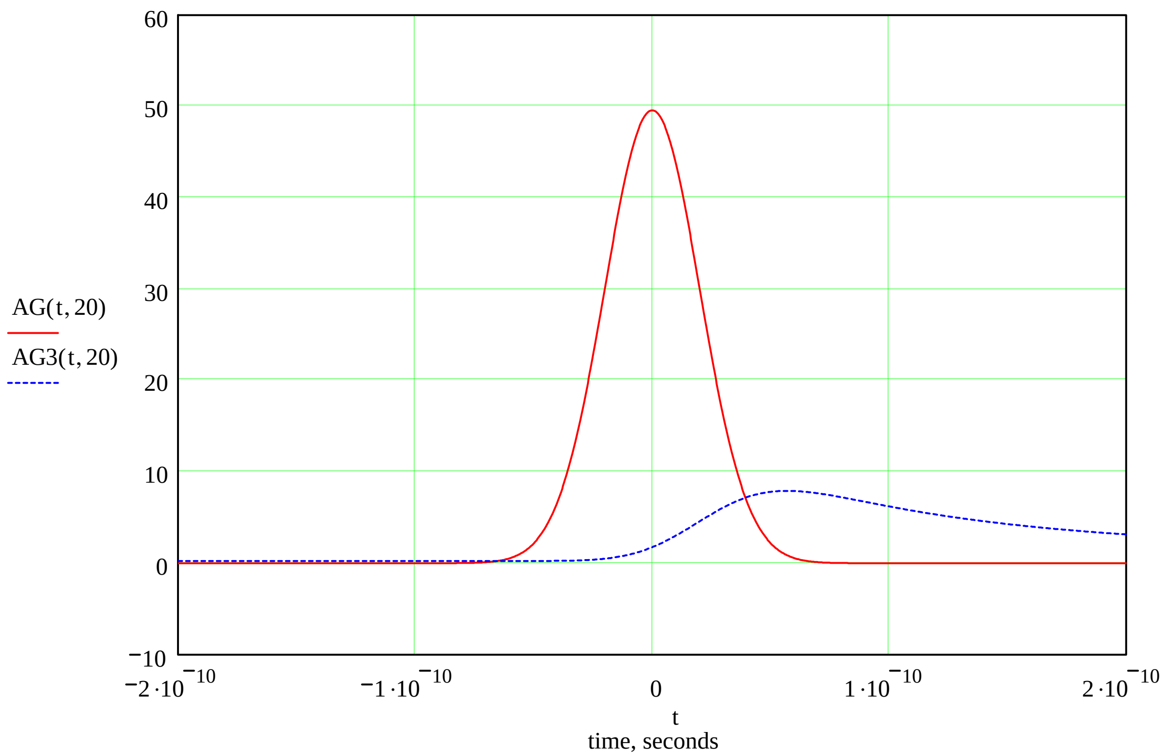
============ Example of cable attenuation and dispersion =============

$atten := 5.0$ attenuation, dB per 100 meters at 402.5 MHz

$alph(n) := \frac{atten}{8.686} \cdot \sqrt{n}$ attenuation (nepers) and phase shift (radians) per 100 meters at harmonic n of 402.5 MHz

Gaussian pulse profile with cable attenuation and dispersion, after 100 meters of lossy coax.

$$AG3(t,s) := 1 + 2 \cdot \sum_{n=1}^{100} \exp(-alph(n))\, g(n,s) \cdot \cos(n \cdot w \cdot t - alph(n))$$



Because the attenuation (due to eddy currents in the conductors) in a coax cable is proportional to the square root of frequency, the dispersion at a particular frequency (measured in radians) is numerically equal to the attenuation (measured in nepers). One neper, equivalent to 1/e of attenuation, is equal to 8.686 dB. This 1:1 relation between nepers and radians is true only when the frequency dependence is $f^{1/2}$.

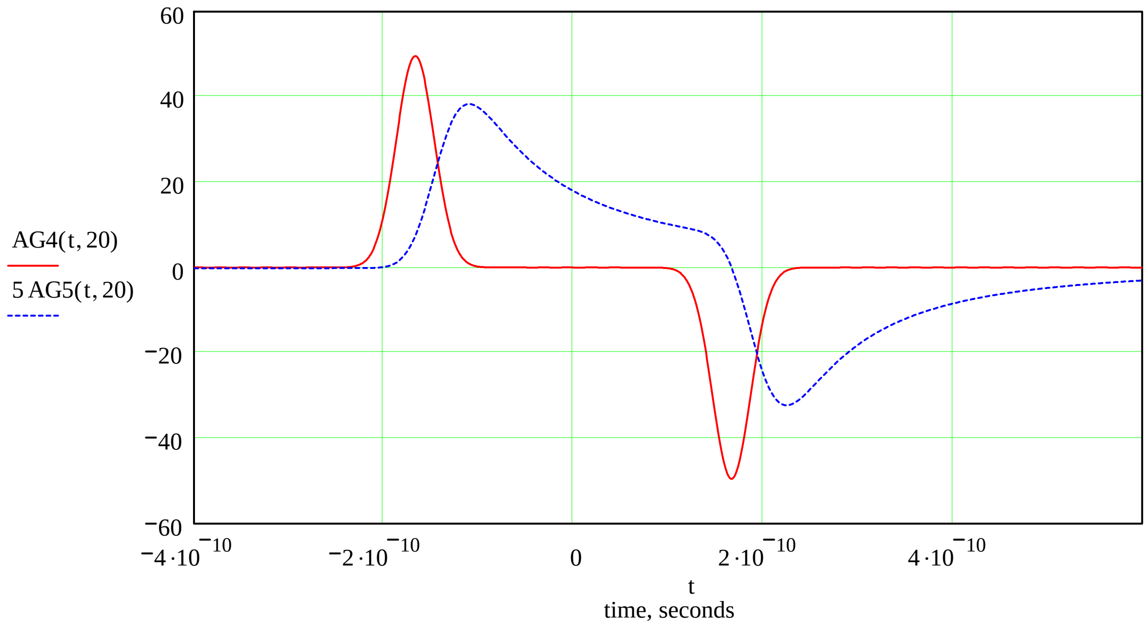
============== Pulse doublet in 5 cm long BPM at beta = 1 ================

$dt := \frac{5}{c}$ $\qquad dt = 1.668 \times 10^{-10}$ transit time, sec, for 5-cm long electrode

$AG4(t,s) := AG(t+dt,s) - AG(t-dt,s)$ doublet without attenuation and dispersion

$AG5(t,s) := AG3(t+dt,s) - AG3(t-dt,s)$ doublet with attenuation and dispersion

note factor of 5 in gain of attenuated signal in plot



==

[1] R. Shafer, Proceedings of the 1989 Beam Instrumentation Workshop (BNL). See page 31.

========== Ratios of full width at base to rms width for parabolic & cosine-squared distributions ===

========= cosine squared ============

$a := \frac{\pi}{2} \qquad R1 := \int_{-a}^{a} x^2 \cos(x)^2 \, dx \qquad R2 := \int_{-a}^{a} \cos(x)^2 \, dx \qquad Ratio := 2 \cdot \sqrt{\frac{R2}{R1}} \cdot a \qquad Ratio = 5.532$

============= parabolic ==================

$a := 1 \qquad R1 := \int_{-a}^{a} x^2 \left(1 - x^2\right) dx \qquad R2 := \int_{-a}^{a} \left(1 - x^2\right) dx \qquad Ratio := 2 \cdot \sqrt{\frac{R2}{R1}} \cdot a \qquad Ratio = 4.472$